

RANDOM WALKS AND FRACTAL STRUCTURES IN AGRICULTURAL COMMODITY FUTURES PRICES

by

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WORKING PAPER 02/02

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October 3, 2001

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Does Ordinary or Fractional Brownian Motion Describe Agricultural Commodity Futures Prices?

Abstract

This paper investigates whether the assumption of Brownian motion often used to describe commodity price movements is satisfied. Using historical data from 17 commodity futures contracts specific tests of fractional and ordinary Brownian motion are conducted. The analyses are conducted under the null hypothesis of ordinary Brownian motion against the alternative of persistent or ergodic fractional Brownian motion. Tests for fractional Brownian motion are based on a variance ratio test and compared with conventional R-S analyses. However, standard errors based on Monte Carlo simulations are quite high, meaning that the acceptance region for the null hypothesis is large. The results indicate that for the most part, the null hypothesis of ordinary Brownian motion cannot be rejected for 14 of 17 series. The three series that did not satisfy the tests were rejected because they violated the stationarity property of the random walk hypothesis.

JEL Classification G0;

Key Words : Random Walk, Fractional Brownian Motion, Futures Prices

1.0 Introduction

The notion of a random walk in futures prices has, for the most part, been treated as an assumption rather than a hypothesis. The assumptions of a random walk and geometric Brownian motion have not only led to closed form solutions for pricing derivatives (e.g. Black Scholes, 1973, Black, 1976 and Merton 1973), but have also provided a simple mechanism for generating derivative prices using Monte Carlo methods (e.g. Boyle, Broadie and Glasserman, 1997). From an economic perspective a random walk has been tied to the concept of efficient and arbitrage-free markets and this has led to many equilibrium models in finance for both traded and non-traded assets, securities and derivatives (e.g. Boyle and Wang 1999, Cox, Ingersoll and Ross, 1985, Garman 1977, or Rubinstein 1979). So critical is the random walk assumption that treating it as a null hypothesis, and rejecting the null hypothesis, has wide spread theoretical and practical consequences for the pricing of derivatives, in particular, and market efficiency in general. Failure to accept the null hypothesis implies that markets can be arbitrated. Persistent arbitrage challenges the risk neutral assumption of classical derivative pricing, which requires that assets grow at the risk-free rate, rather than the natural growth rate.

Rejection of the null hypothesis of ordinary Brownian motion as a descriptor of price movements gives consideration to an alternative hypothesis of fractional Brownian motion. The term fractional Brownian motion (fBm) was coined by Mandelbrot and Van Ness (1968) to describe particular random series that do not display self-similar behaviour. These random series have a memory characterized by strong interdependence between current and future samples. Because sub samples of data tend to be correlated the data violate the Markovian assumption of independence, and fBm is not a semi martingale (Rogers, 1997). We care about Markov processes and the martingale measures because these are crucial in the pricing of traded or non-traded financial or commodity derivatives under the no arbitrage (e.g. Black 1976, Black and Scholes 1973) or equilibrium models (Cox, Ingersoll and Ross 1985, Garman 1977, Rubinstein 1979). Indeed, Rogers (1997) shows that a fBm can give rise to arbitrage opportunities in general, while Cutland, Kopp and Willinger (1995) and Sottinen (2001) show that a fractional Black-Scholes model, in particular, gives rise to arbitrage opportunities. For example, defining H as a self similarity parameter in the sense of Hurst (1951) and Mandelbrot and Van Ness (1968), then $dZ = \varepsilon \sqrt{t^{2H}}$ is a Wiener process for $H=0.5$, where t a time increment, and ε is a standard normal variate with mean zero and variance equal to one. When $H=0.5$ the Ito property

that $dZ^2 = t$ is satisfied. However, if $H \neq .5$ then $dZ^2 = t^{2H}$ and Ito's lemma does not hold. Loosely speaking, the order of bias in options pricing is determined by the implicit assumption that $t' = t^{2H}$, and Ito's lemma is applied for $dZ^2 = t'$, an obvious falsification for $H \neq .5$. By simple substitution, the Black-Scholes call option formula for an asset with current value X , strike price E , and time step t , would become

$$C(X,t') = X N(d1) - E e^{-rt} N(d2)$$

with $N(\cdot)$ representing the cumulative normal distribution function evaluated at $d1$ and $d2$,

$$d1 = (\log(X/E) + (r + .5 \sigma^2) t^{2H}) / \sigma t^H$$

and

$$d2 = (\log(X/E) + (r - .5 \sigma^2) t^{2H}) / \sigma t^H .$$

For $H=.5$ this reduces to the usual Black-Scholes formula. All other things held constant, $\partial C(X,t')/\partial H > 0$ over the defined range of H , $0 < H < 1$. In other words the value of the call option will be higher or lower than the implied Black-Scholes price for $H > .5$ and $H < .5$ respectively. Under the fractional Brownian motion assumption of Mandelbrot and Wallace (1968), values of $H > .5$ imply long range or persistent memory. As H rises above $.5$ there is increasing positive dependence between time steps so that there is a greater chance that the call option will expire in-the-money. As H falls below $.5$, the time series exhibits persistent reversals due to a short memory process. Time steps are negatively correlated. The lower the value of H the more jagged the time series will appear and the more likely that a reversal will occur. Since an increase in the price will likely be followed by a reversal there is a greater chance that the call option will expire out-of-the-money, so the value of the call option will fall relative to the ordinary Brownian motion case.

The arbitrage opportunities now become obvious. Consider the call option as a function of H , $C(X,t,H)$. For a persistent time series, $C(X,t,H > .5) > C(X,t,H=.5)$ so a knowledgeable speculator anticipating that the likelihood of the option expiring in-the-money is greater than that suggested by Black-Scholes could buy an option priced at $H=.5$ immediately and sell the option at expiry for an expected gain of $E[C(X,t,H > .5) - C(X,t,H=.5)] > 0$. Likewise for $H < .5$ a knowledgeable speculator, anticipating that the likelihood of an expiry in-the-money is lower than that represented by Black-Scholes, could sell an option for an expected capital gain of $E[C(X,t,H=.5) - C(X,t,H < .5)] > 0$. That speculators can use historical price patterns to infer, with some degree of success, future price patterns, violates the most basic tenets of the efficient

market hypothesis. Furthermore, that arbitrageurs can leverage fractional Brownian motion to obtain (expected) profits above the market price of similar-risk assets violates the no-arbitrage assumption of options pricing.

The subject of this paper is the random walk generated by Brownian motion. The objective of this paper is to determine if commodity futures prices satisfy the Brownian motion assumption. In the alternative, we are interested in determining if Brownian motion is fractal, (i.e. fractional Brownian motion). Recent investigations are mixed on the existence of memory in financial and commodity price series. In the broader context of existence of long-term memory, Comte and Renault (1996) prove that the stochastic differential equations commonly employed to characterize financial and commodity price movements can be extended to encompass long memory models. Hommes (2001) suggests that nonlinearities in financial time series could be due to heterogeneous expectations among traders that give rise to adaptive belief systems. Several researchers have investigated fractal structure in futures contracts using a procedure from Hurst and Mandelbrot (1965, 1972) (R-S analysis) and/or a modified Hurst procedure owing to Lo (1991). Studies using R-S analysis have shown that there is persistent ($H > .5$) behaviour in financial markets (Greene and Fielitz (1977), Booth, Kaen, and Koveos (1981, 1982a, b , Peters 1996) and futures markets (Helms, Kaen and Rosenman 1984, Barkoulas, Labys and Onochie 1997, Corazza, Malliaris, and Nardelli, 1997, Peters 1996, Cromwell, Labys and Kouassi 2000). However, recent research using Lo's (1991) modification for correlated bias fails to reject the null hypothesis of no fractal structure in futures prices (Crato and Ray 2000).

This study differs from previous studies in that I test specifically for Brownian motion and note that much of the fractal research conducted on economic data is geared towards determining if long or short term memory exists, and from there making inferences about random walks. Noting that with fractional Brownian motion a time series can be stationary while failing the independence assumption, Brownian motion is examined using a time series stationarity test in addition to a variance ratio test. The value of H is estimated directly from the variance ratio. In addition, the value of H is also estimated, for comparison purposes, from conventional R-S analysis.

Failure to reject the null hypothesis of no fractal structure using the variance ratio test, or R-S analysis, does not unto itself imply a random walk (Peters 1996) unless the stationarity conditions are also satisfied. Likewise, rejecting the null hypothesis of a random walk by itself

does not necessarily imply the existence of fractals. This paper takes the position that a true test of Brownian motion is a joint test of stationarity and fractal structure. If stationarity is true and the variance ratio test indicates $H = .5$, then agreement would confirm an ordinary Brownian motion. If stationarity is true and $H \neq .5$, then the random walk follows a fractional Brownian motion. If the time series is non-stationary in differences then the null hypothesis of a random walk, and hence any form of Brownian motion, is rejected.

The specific hypotheses are examined for 17 time series of futures prices of approximately 950 days. Statistical tests are based on Monte Carlo simulations of the standard errors of particular parameter estimates for the stationarity and variance tests. The Monte Carlo simulations indicated that the standard errors on stationarity and variance parameters estimates were high. Simulated values of H for example had a mean of $.5$ and a standard deviation of $.125$. All of the time series indicated ordinary Brownian motion under the null hypotheses. Tests of stationarity indicated that 3 of 17 time series did not satisfy the stationarity assumption. The conclusion is that most time series are consistent with ordinary Brownian motion - the time series are not fractal. The 3 futures contracts that did not satisfy the random walk assumption were rejected because of the stationarity conditions rather than persistent or anti-persistent behaviour. However, a qualitative assessment of the H values from both the variance ratio test and R-S analysis suggests that most futures contracts show anti-persistent or mean-reverting patterns.

The paper proceeds as follows. The next section explains the variance ratio test for a random walk and introduces the concepts of Brownian and Fractional Brownian motion. This is followed by a review of some basic concepts in chaos theory, fractals, and the R-S test. Both the variance ratio and R-S tests are applied to 950 daily observations of futures prices for 17 commodities traded on the Chicago Mercantile Exchange, Chicago Board of Trade, and Winnipeg Commodities Exchange. The results are discussed and the paper is then concluded.

2.0 Fractional Brownian Motion

In the classical models of random walk, efficient markets, and the pricing of derivatives it is assumed that the percentage change in the futures price over a discrete interval of time is governed by

$$(1) \quad dF = \alpha F dt + F \sigma_F dZ$$

where $dZ = \varepsilon\sqrt{t}$ is a Gauss - Wiener process, F is the futures price, α is the instantaneous change in futures prices and σ is the variance of the percentage change in futures prices.

A fractional Brownian motion in contrast is specified by

$$(2) \quad dF = \alpha F dt + F \sigma_F dX$$

where $dX = \varepsilon\sqrt{t^{2H}}$. In (1) and (2) the term ε can be interpreted as a random shock over the prescribed time interval. In (2) the parameter H measures the fractal dimension of the stochastic process. It is analogous to the Hurst coefficient in standard R-S analysis. H can take on any value between 0 and 1, and the fractal dimension is measured by

$$(3) \quad D = 2 - H.$$

The variable D has particular meaning. Consider a two-dimensional plain such as a piece of paper. A straight line drawn on the paper is of dimension 1. If the paper is completely shaded then it is of dimension 2. Therefore a value of $H = 0$ represents a dimension of 2 and $H = 1$ represents a line of dimension 1. A random walk is neither a plane or a line but something in between. A pure random walk has $H = .5$ and a biased random walk has $H \neq .5$. For $H > .5$ the system is said to be persistent. Persistence refers to a stochastic system that has long-term memory such that an event at some point t is positively correlated with observed events at some future period, $t + \Delta t$. It implies positive correlation among random events over time. The limit of this is a straight line representing a perfectly correlated system.

In contrast, $H < .5$ becomes increasingly jagged. It would appear on a piece of paper, to be more volatile and more erratic than a pure random walk with $H = .5$. The system also reverses itself frequently and for this reason it is said to be anti-persistent, ergodic, or mean reverting. It is also referred to as a short-memory process and unlike a persistent system it is characterized by negative correlation. That is, an event at some moment in time t (say an increase in futures price) will cause a reversal at some point in the future at $t + \Delta t$.

We are concerned with the properties of dX and will examine them using results from Crownover (1995). For this purpose define $dX = X(t_2) - X(t_1)$ with expected value of zero and variance $\sigma^2(t_2 - t_1)^{2H}$. A fractal Brownian motion has a Gaussian distribution of the form

$$(4) \quad \Pr(dX < x) = \frac{1}{\sqrt{2\pi}\sigma(t_2 - t_1)^H} \int_{-\infty}^x \exp(-1/2(\frac{\mu}{\sigma(t_2 - t_1)^H})^2) d\mu$$

If $H = .5$ then a fractional Brownian motion is the same as standard Brownian motion as used in equation (1). Likewise the variance of a fractal Brownian motion

$$(5) \quad E[X(t_2) - X(t_1)]^2 = \sigma^2(t_2 - t_1)^{2H}$$

reduces to that of standard Brownian motion when $H = .5$,

$$(6) \quad E\{X(t_2) - X(t_1)\}^2 = \sigma^2(t_2 - t_1).$$

The critical difference between (6) and (5) is that the variance property for standard Brownian motion increases linearly in time, whereas the variance in fractal Brownian motion is a non-linear function of H . From (5)

$$(7) \quad \partial E[X(t_2) - X(t_1)]^2 / \partial H = 2\sigma^2 \ln(t_2 - t_1)(t_2 - t_1)^{2H}$$

is positive for any H and $t_2 - t_1 > 1$, and

$$(8) \quad \partial^2 E[X(t_2) - X(t_1)]^2 / \partial H^2 = (2\sigma^2 \ln(t_2 - t_1)(t_2 - t_1)^H)^2$$

is also positive. In other words, a fractional Brownian motion is characterized by a variance increasing in H at an increasing rate. The implications for arbitrage and derivative pricing are evident: an option priced according to ordinary Brownian motion, a priori, will be overpriced for $H < .5$, precisely priced for $H = .5$, and under priced for $H > .5$.

In part, the difference between variance measured by ordinary and fractional Brownian motion is due to correlation between time increments. Under ordinary Brownian motion the Markovian property of independence between time increments is required, but fractional Brownian motion does not, as shown below, exhibit this property.

To see this define the covariance between any two time increments, as

$$(9) \quad \text{COV}(t, \Delta t) = E[X(t) - X(0)] [X(t + \Delta t) - X(t)].$$

Taking the standard deviation of (5) and substituting it into (9) gives

$$(10) \quad E[X(t) - X(0)] [X(t + \Delta t) - X(t)] = .5 \sigma^2 [(t + \Delta t)^{2H} - t^{2H} - \Delta t^{2H}]$$

Once again by setting $H = .5$ the right hand side of (10) collapses to zero and the independent increments assumption is satisfied. For any $H \neq .5$, it is not satisfied. As H approaches zero the limit of covariance approaches $-.5\sigma^2$, and as H approaches $+1$, covariance approaches $\sigma^2(t \Delta t) > 0$.

Taking the derivative of (10) with respect to the time interval Δt yields

$$(11) \quad \partial \text{COV}() / \partial \Delta t = \sigma^2 H ((t + \Delta t)^{2H-1} - \Delta t^{2H-1})$$

For $H < .5$ the covariance term decreases with increasing time steps. Hence the term 'short memory'. In contrast the term 'long memory' comes from the results that covariance increases with increased time steps when $H > .5$. Importantly, since the independence property is violated, a fractional Brownian motion is not a Markov process, nor is it a Martingale.

Finally, even though a fractional Brownian motion does not satisfy the independence property it does satisfy the stationarity in differences property. That is, since variance depends only on the difference between t_1 and t_2 , and not t_1 and t_2 particularly, the increment dX is stationary. We can use the stationarity and independence assumptions to test for fractional versus ordinary Brownian motion in the following way. Under the null hypothesis of a stationary time series a specification test that rejects the null automatically eliminates the time series as being either ordinary or fractional Brownian motion, let alone a random walk. Failure to reject the null hypothesis is sufficient to conclude a random walk, but is not sufficient on its own to declare a Brownian motion. The test for ordinary or fractional Brownian motion is applied only to stationary time series. The specification test is conducted under the null hypothesis $H_0: H=.5$. Failure to reject the null implies an ordinary Brownian motion. Rejecting the null implies $H \neq .5$ and a persistent or antipersistent fractional Brownian motion would be concluded for $H > .5$ and $H < .5$ respectively.

3.0 Random Walks and the Variance Ratio Test

Lo and Mackinnon (1999) used the variance property of Brownian motion to test for random walks. The essence of their argument is that the variance of any step or lag K ($1 < k \leq T$) must be a linear multiple of the variance of a single step or lag. For example, the variance of price changes over a 2 day period will be twice the variance of the change in 1 day or the variance over 20 days will be twenty times the variance of 1 day. Hence

$$(12) \quad \sigma_K^2 = K\sigma_1^2$$

and

$$(13) \quad VR = \frac{\sigma_K^2}{K\sigma_1^2} = 1.$$

The result suggests a specific test for a random walk. First, calculate the percentage change in prices for each of $\ln(F_{t+1}) - \ln(F_t)$, allowing for overlapping prices. Second calculate the

variance, $\text{VAR} (\ln(F_{t+k}) - \ln(F_t))$, for each k step including $k = 1$. Third, divide the calculated variance for each $k \geq 1$ by the variance for $k = 1$. This gives k ratios of the form σ_k^2 / σ_1^2 .

The results allow for a number of tests. The Lo and Mackinnon (1999) approach is to treat each of the k ratios as a separate hypotheses. That is

$$(14) \quad H_0: \frac{\sigma_k^2}{\sigma_1^2} - k = 0 \quad \forall k.$$

Lo and Mackinnon (1999) provide a formula for calculating the asymptotic variance of the ratio and provide a standardized test for the null hypotheses. In the alternative, an equivalent test would be to regress

$$(16) \quad \sigma_k^2 / \sigma_1^2 = \hat{a} + \hat{b}k + \varepsilon$$

and set $H_0: \hat{b} = 1$. Failure to reject H_0 would indicate that variance increases linearly in time as required by the random walk hypothesis.

In the context of fractional Brownian motion the above model may not be specific enough since its variance is given by $\sigma^2 T^{2H}$. To test for a biased random walk follow the steps described above for calculating variance ratios. To test for fractional Brownian motion we need an estimate of the H coefficient. The following regression can be used to estimate the value for H .

$$(16) \quad \ln(\sigma_k^2 / \sigma_1^2) = \alpha_0 + \alpha_1 \ln(k) + \varepsilon$$

with $H_0: \alpha_0 = 0$ and $H_0: \alpha_1 = 1$. In (16) the value of H can be calculated from $\alpha_1 = 2H$ or $H = \alpha_1 / 2$. If $\alpha_1 = 1$ then $H = .5$ and there is no evidence of fractal structure. If $\alpha_1 > 1$ then $H > .5$ and this would indicate long-term memory and positive autocorrelation. If $\alpha_1 < 1$ then $H < .5$, memory is short and the system is ergodic or mean reverting.

4.0 Stationary Increments in Futures Prices

Even though fractional Brownian motion does not satisfy the property of independent increments, it still must satisfy the Gaussian assumption of stationary increments. In general stationary increments imply that the first difference of the returns series are independent for any choice of t . that is the differenced series

$$\{X_2 - X_1, X_3 - X_2, X_4 - X_3 \dots X_t - X_{t-1}\}$$

are independent. Furthermore the process is stationary across any time step. This means that for a time step k (where k can equal days or weeks, etc.)

$$(17) \quad E[x_k - x_1] = E(x_k - x_{k-1} + x_{k-1} - x_{k-2} + \dots + x_2 - x_1) \\ = k\mu$$

where $\mu = E[x_t - x_{t-1}]$ across all t . This definition of stationarity states that the k -step difference between any two observations is a linear function of the mean 1-step difference. Hence

$$(18) \quad E\left[\frac{x_{t+k} - x_t}{x_{t+1} - x_t}\right] = k.$$

This leads to a simple test for stationarity by estimating the following regression,

$$(19) \quad \left[\frac{x_{t+k} - x_t}{x_{t+1} - x_t}\right] = \beta_0 + \beta_1 k + e.$$

Under the null hypotheses H_0 : $\beta_1 = 1$ the linearity assumption, and hence a finding of a stationary process, will be rejected if H_0 is rejected. The alternative hypothesis H_A : $\beta_1 \neq 1$ implies that the increments are non-stationary.

The test is slightly more complicated to generate than standard unit-root tests since the numerator in (19) requires calculating the differences for each step. For example if $k = 150$ then the values $x_{t+150} - x_t$, $x_{t+149} - x_t$ and so on must be calculated. More degrees of freedom will be available if overlapping rather than non-overlapping increments is used. On the other hand, the procedure is consistent with the unit-root test since by using $x_{t+1} - x_t$ in the denominator an AR(1) process is implicitly assumed.

5.0 Classical R-S Analysis

A common approach to measuring fractal dimension uses the non-parametric approximation developed by Hurst (1951) to measure randomness in water flows. The approach developed by Hurst is called R-S analysis or the rescaled range analysis. Nonetheless, it is apparent that Hurst's R-S analysis is commonly used by researchers searching for fractal structure in financial instruments. It is, therefore, important to compare the measure of fractal structure using the variance ratio measure in equation (16) to the classical Hurst measure.¹

¹ Although Lo's (1991) modification has been used by a number of fractal investigators (e.g. Corazza et al., 1997; Crato and Ray, 2000) it is not required here. Lo's (1991) modification is required only in the case where R-S analysis

The R-S procedure is described in an economic context by Mandelbrot (1972), Mandelbrot and Wallis (1969), Peters (1996), Lo (1991), and Helms, Kaen and Rosenman (1984) among others. The procedure uses a time series data set comprised of T observations. The data set is then divided into N non-overlapping sub-series of length k (N equals the lower integer of T/k). From an initial starting point k , the procedure is repeated by increasing k in fixed increments, $k_2 = k_1 + \mu$, $k_3 = k_1 + 2\mu$ and so on. For each incremental k the following procedure is taken.

For each $n = 1, N$ sub-series calculate the mean $\bar{x}_n = \frac{1}{k} \sum_{i=1}^k x_{in}$. Next define

$$(20) \quad W_{k,n} = \sum_{i=1}^k (x_{i,n} - \bar{x}_n) \quad n=1, N$$

For each sub-series n , the partial sums are calculated and for each addition in $W_{k,n}$ the value of $W_{k,n}$ will increase or decrease. There are $i=1, k$ partial sums (e.g. $(x_{1n} - \bar{x}_n)$ is the first partial sum, $(x_{1n} - \bar{x}_n) + (x_{2n} - \bar{x}_n)$ is the second partial sum and so on) and the minimum and maximum values of the k partial sums for each n are recorded.

Using the minimum and maximum values define the range

$$(21) \quad R_{k,n}^* = \text{Max}(W_{k,n}) - \text{Min}(W_{k,n}) \quad n=1, N.$$

The range is then rescaled to the standard deviation of the sub-series sample,

$S_n = \left(\frac{1}{k} \sum_{i=1}^k (x_{in} - \bar{x}_n)^2 \right)^{1/2}$, to get the classical R-S statistic

$$(22) \quad R_{k,n} = \frac{\text{Max}(W_{k,n}) - \text{Min}(W_{k,n})}{S_n} \quad n=1, N$$

There are $N=T/k$ range statistics calculated for each k (with N decreasing with an increase in k) so the final step is to average the range statistics

$$(23) \quad \bar{R}_{kn} = \frac{1}{N} \sum_{n=1}^N R_{k,n}.$$

There will be one value of \bar{R}_{kn} for each k iterant. Under the null hypothesis that the rescaled range is asymptotically related to the sub-series size, Mandelbrot and Wallis (1969) following Hurst (1951) suggest

indicate long run persistent dependence. The correction is to remove any short run memory effects from the long-run memory effects. Since the results do not indicate long-memory, the modification is not required.

$$(24) \quad \bar{R}_k \cong ak^H$$

where H (the Hurst coefficient) is the same theoretical measure of fractal dimension associated with $\sigma\sqrt{t^{2H}}$ from the fractal Brownian motion process. By estimating the value \bar{R}_k for each k, the coefficient H can be estimated by the least squares regression

$$(27) \quad \ln(\bar{R}_k) = \ln(a) + H \ln(k) + e.$$

The value of H from this R-S analysis can then be compared to the value of H estimated from the variance ratio.

6.0 Data

Seventeen futures contracts for agricultural commodities are examined for Brownian motion. Summarized in Table 1, the data represent 950 matched daily observations from 1996 through February 7, 2001 on the nearby futures price. The futures contracts include grains and oilseeds, livestock and livestock products, and cocoa, coffee, orange juice and sugar. The contracts are traded on the Chicago Mercantile Exchange (CME), the Chicago Board of Trade (CBOT), the Coffee, Sugar and Cocoa Exchange (CSCX) and the Winnipeg Commodity Exchange (WCE). Alberta barley, rapeseed, Winnipeg oats and Winnipeg wheat are denominated in Canadian dollars while others are in \$U.S.

The sample means and range are given in Table 1. In the last two columns the annualized geometric growth rate and volatility based on a 250-day trading year are presented. The results show that 13 of 17 commodities faced price declines over this period with the largest declines being on CBOT and WCE oats at -19.7% and -22.7% respectively. Feeder cattle (CME) showed the largest annual gain of approximately 10.2%/year.

On average, volatility exceeded 30% per year. The most volatile commodity was pork bellies (CME) at 55.2% followed by coffee (CSCX) at 52.8%, lean hogs (CME) at 42.6% and wheat (CBOT) at 40.1%. Sugar (CSCX) was the least volatile at only 8.5% and Alberta barley (WCE) had the second lowest volatility at 19.4%.

7.0 Results

The methods employed were directed towards testing the single null hypothesis that futures prices followed a random walk as described by geometric Brownian motion. Failure to

reject the null hypothesis will result if it is shown that the price differences are stationary and $H = .5$. If $H \neq .5$ then this would be evidence of a fractional Brownian motion. If the time series is non-stationary then the time series cannot be either fractional or non-fractional Brownian motion.

The parametric tests of stationarity and H are based on a lag structure with $k = 150$ days. Initial estimates were done for up to 300 lags but following Peter's (1996) suggestion logarithmic plots of R/S statistic and k were constructed, and these plots indicated that any memory effect dissipated by about 150 days for many of the futures contracts. Hence all estimation is done for $k = 150$ days.

7.1 Standard Error Estimation

As indicated previously least squares estimation of the slope coefficients in equations (16) and (19) would not provide asymptotic standard errors since each of the price series represents but a single stochastic path of many potential paths that could arise under a random walk hypothesis. To overcome this problem a Monte Carlo technique was used to estimate the standard errors. This technique generated a 1,000-day path with zero mean and volatilities of .10, .20, .30, .40, .50, and .60. The Monte Carlo model was constructed using the exact estimation process used on the 950-day actual series to estimate β_1 (equation 19) and H values ($H = \alpha_1/2$ from equation 16) except that the simulated results were done for only 40 lags.² The experiment was replicated 2,500 times. The results provided a mean value for H approximately equal to .5 with a standard deviation about the estimate \hat{H} being .125. This standard error was stable for all volatility measures.

The standard errors for the estimate of $\hat{\beta}_1$ were more variable ranging from .78, .77, .75, .78, .76, and .70 for volatilities of .10 through .60 respectively. With an average volatility across all contracts of .30 an estimated standard error of .75 is used to establish the asymptotic confidence limits about $\hat{\beta}_1$.³

² Using 40, rather than 150, lags provided standard errors that would be lower than those from 150 lags (see Fama and French, 1988). A lower standard error implies a smaller confidence interval around the expected value. This provides a more rigorous test of the null since it decreases the acceptance region in favour of the rejection region.

³ Each simulation was run using the same seed value for the Monte Carlo draws. The simulations were run for several different seed values and similar results were obtained.

The Monte Carlo estimates of asymptotic standard errors fails to reject the null hypothesis of stationarity at the 95% level if $-0.47 \leq \hat{\beta}_1 \leq 2.47$ and the null hypothesis $H_0: H = .5$ should not be rejected at the 95% level if $0.255 \leq \hat{H} \leq 0.745$.

The standard errors in both instances are large and qualitatively important since they suggest that a finding of $\beta_1 \neq 1$ or $H \neq .5$ cannot be taken as a rejection of the random walk hypothesis as a matter of course. Any single sequence of actual data can, by chance alone, have $\beta_1 \neq 1$ and $H \neq .5$, and still be consistent with a Brownian motion assumption. The range over which the standard error rejects the null hypothesis of ordinary Brownian motion is small ($0 \leq \hat{H} < .255$ or $\hat{H} > .745 \geq 1.0$). Likewise rejection of the null hypothesis on stationarity requires estimates of $\hat{\beta}_1 < -0.47$ and $\hat{\beta}_1 > 2.47$.⁴

Estimates of actual values of $\hat{\beta}_1$ are found in Table 2, for the entire period (column 2) and four sub periods of 180 days with the first sub period representing days 1-179 and so on. The hypothesis test is directed toward the full sample estimates in column 1, while columns 3-5 are presented to illustrate the range of variance about $\hat{\beta}_1$ in smaller samples.

Under the null hypothesis only three series fall outside of the acceptance region. These are coffee ($\hat{\beta}_1 = -.102$), lean hogs ($\hat{\beta}_1 = -2.89$) and pork bellies ($\hat{\beta}_1 = -1.02$). Since stationarity is a necessary condition for a Gaussian and Brownian process these three commodities can immediately be eliminated from the set of prices displaying geometric Brownian motion (even though all of them have at least one sub sample sequence that might satisfy stationarity).

The remaining 14 commodities will have either ordinary or geometric Brownian motion at the 95% confidence level if $.255 \leq \hat{H} \leq .745$. The estimates of \hat{H} are presented in Table 3. In Table 3, column 2 provides the estimate of H used in the hypothesis test and columns 3 through 5 show the four 180-day sub periods. Column 6 provides an estimate of H using R-S analysis as a point of comparison. Since no value of \hat{H} falls outside of the asymptotic 95%

⁴ I have not found previous research that supported Monte Carlo estimates of the asymptotic standard deviations of H and β_1 . However, in Fama and French (1988) a similar approach is used to estimate the standard errors of first order autocorrelation coefficients. Qualitatively they are able to support the conjecture that stock price movements have stationary and random components. However, when their specific tests were assessed using standard errors from Monte Carlo simulations they found that the null hypothesis (of non-stationarity in prices) was difficult to reject. In fact they speculate that the large standard errors in a pure random walk may make such hypotheses altogether untestable (Fama and French, 1988, page 257). A wide acceptance region for unit roots in time series data has also been discussed by Kwiatkowski et al (1992).

confidence limit there is no instance where the estimated value of H is statistically different from .5. From a statistical perspective ordinary Brownian motion cannot be rejected for all series except coffee, lean hogs and pork bellies.

In a qualitative sense, accepting the values as given has several implications. First, with the exception of sugar which shows a slightly persistent dynamic with $H = .543$, the evidence suggests that commodity futures prices are ergodic or mean-reverting. This observation is in opposition to recent concerns regarding persistent long-term memory in commodity futures contracts (Barkovlas et al. 1997, Corazza et al., 1997 or Crato and Kay 2000). The results in Table 3 provide no support for long-term memory.

The R-S estimates in Table 3 are different than those presented in column 1. Qualitatively, corn, fluid milk, pork bellies, soybeans, sugar and Winnipeg oats display persistent tendencies with $H > .5$. However only Winnipeg oats (.566) and perhaps fluid milk (.540) are sufficiently higher than .5 to warrant concern.

The remaining 11 commodities have R-S estimated $H \leq .5$. Winnipeg wheat (.499) and rapeseed (.494) are virtually identical to .5 and would thus be characterized as having a pure random walk. The remaining futures prices again display mean-reverting tendencies. Qualitatively the main conclusion is that the Mandelbrot-Hurst approach provides results that are not inconsistent with the variance ratio approach.

8.0 Implications of Results

The results of this study have significant implications for the analysis of futures (and other financial) time series. The evidence of this paper is that the null hypotheses of stationary increments and $H = .5$ cannot generally be rejected. These two conditions are sufficient to support the conjecture that futures prices follow a random walk in general and an ordinary Brownian motion in particular,. A comparative analysis of the variance ratio test and the unadjusted (in the sense of Lo) Hurst-Mandelbrot R-S statistic also provides support for the conclusions reached.

While these null hypotheses are not rejected on the basis of simulated standard errors a qualitative assessment of the results suggests, in general, that futures prices series are mean reverting. This conclusion is consistent with recent findings by Corazza et al. (1997) and Crato and Ray (2000) and is at odds with earlier findings by Helms et al. (1984) and Barkoulas et al.

(1997). Qualitatively, the variance ratio tests indicate that most commodities have ergodic or mean reverting properties. This is a comforting result since it indicates that in the long run the laws of supply and demand work in a classical dimension (e.g. the cob-web cycle). An opposing result would suggest that supply and demand are complementary; an increase in demand leads to an increase in supply, which leads to an increase in demand and so on. While such an economy may operate in the short-run a persistent time series would suggest that it goes on indefinitely, that demand is never saturated, and that equilibrium is never attained.

Only two series showed some form of a long-term memory, but the measured persistence is not qualitatively different than .5. Nonetheless it is interesting to note that one of the persistent series was the Milk futures contract. Up to January 2000 this contract was based on the U.S. base formula price (BFP), and was settled monthly. The settlement price was not based on observed market transaction as is normally the case in price discovery. Rather the settlement price was based on a monthly survey of U.S. processors by the U.S.D.A. Only on the settlement data was there price discovery and even so, the survey was limited to a small proportion of processors. In the absence of transparent price discovery it is not surprising that hedgers would bid a futures price based on past performance.

From an analytical perspective this paper has provided a means to empirically test for fractional Brownian motion using variance ratios. This is a parametric approach that relies on the fractional definition of the Wiener process. In contrast, the Hurst-Mandelbrot approach is non-parametric. Given the qualitatively similar results, this is not necessarily a criticism of the Hurst-Mandelbrot approach, but an approach to measuring fractals and fractal dimension using a consistent-theoretical structure has its advantages. From a computational perspective the approach was less cumbersome than the R-S approach.

Finally, the overall intent of this paper was to determine if commodity futures prices followed a random walk process consistent with non-fractal Brownian motion. The results indicate that futures price movements are consistent with Brownian motion. One of the beneficial outcomes is that, for the most part, the assumption of Brownian motion used in the pricing of options on futures is justified. If Brownian motion is consistent with the efficient market hypothesis (an inference that is, according to Lo and Mackinnon (1999) and Corazza et al. (1997), debatable) then the results of this study indicate that markets are indeed efficient.

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Table 1 Sample Statistics for Futures Price Series

contract	Exchange	Mean	Variance	Standard Dev.	Maximum	Minimum	Geometric Mean	volatility
Alberta Barley price	WCE	137.60	476.14	21.82	196.80	108.50	-0.108	0.194
coffee price	CSCX	129.15	996.54	31.56	261.00	81.35	-0.028	0.528
cocoa price	CSCX	1337.50	62310.14	249.62	1762.00	763.00	-0.116	0.274
corn price	CBOT	272.33	5928.11	76.99	548.00	178.50	-0.146	0.373
Feeder Cattle price	CME	71.50	65.10	8.07	86.88	47.65	0.102	0.208
Fluid Milk price	CME	13.51	4.75	2.18	21.70	9.47	-0.074	0.342
Lean Hogs price	CME	60.25	221.06	14.87	90.12	25.22	-0.050	0.426
live cattle price	CME	65.61	10.49	3.24	73.63	54.80	0.023	0.211
oats price	CBOT	148.85	1755.47	41.89	286.00	99.00	-0.197	0.377
orange juice price	CSCX	97.57	292.75	17.11	138.00	66.80	-0.126	0.395
Pork Bellies price	CME	63.98	249.857	15.80	104.475	32.75	0.069	0.552
Rapeseed canola price	WCE	376.71	3563.54	59.69	490.20	251.30	-0.137	0.213
Soybeans price	CBOT	637.02	15992.26	126.46	894.25	410.00	-0.095	0.277
Sugar price	CSCX	21.78	1.66	1.29	23.09	16.55	-0.070	0.085
wheat price	CBOT	345.37	9079.49	95.28	716.50	224.00	-0.156	0.401
Winnipeg oats price	WCE	121.87	1906.97	43.66	243.00	83.00	-0.227	0.297
Winnipeg Wheat price	WCE	165.08	998.46	31.59	293.40	121.70	-0.128	0.235
Average							-0.067	0.297

Table 2: Estimated Values of Hurst Coefficient from Equation (16) where $H=\alpha_1/2$ for Days in Sample and from R-S Calculations from Equation (27). Null Hypothesis $H_0 = .5$ Accepted within range $0.255 \leq H \leq 0.745$

Days/Contract	940	760	580	400	220 R-S	Hurst
Alberta Barley price	0.414	0.431	0.451	0.428	0.333	0.489
coffee price	0.402	0.441	0.448	0.376	0.065	0.467
cocoa price	0.465	0.431	0.280	0.291	0.117	0.446
corn price	0.348	0.363	0.362	0.363	0.254	0.503
Feeder Cattle price	0.401	0.407	0.360	0.099	0.059	0.461
Fluid Milk price	0.481	0.489	0.473	0.523	0.381	0.540
Lean Hogs price	0.438	0.388	0.342	0.203	0.156	0.486
live cattle price	0.272	0.269	0.243	0.256	0.088	0.460
oats price	0.348	0.321	0.323	0.356	0.194	0.436
orange juice price	0.458	0.479	0.514	0.248	0.141	0.425
Pork Bellies price	0.381	0.356	0.380	0.253	0.206	0.519
Rapeseed canola price	0.396	0.336	0.352	0.251	0.267	0.494
Soybeans price	0.332	0.331	0.324	0.269	0.226	0.515
Sugar price	0.543	0.285	0.261	0.200	0.029	0.519
wheat price	0.231	0.221	0.171	0.168	0.123	0.483
Winnipeg oats price	0.481	0.477	0.459	0.480	0.200	0.566
Winnipeg Wheat price	0.341	0.349	0.345	0.334	0.265	0.499

Table 3: Estimated Values of β_1 From Equation (19): Test for Stationarity. Null Hypothesis $H_0: \hat{\beta}_1 = 1$ accepted in range $-0.47 \leq \hat{\beta}_1 \leq 2.47$.

Days/Contract	940	760	580	400	220
Alberta	1.323	1.277	1.135	3.436	1.462
Barley price					
coffee price	3.084	0.513	8.608	3.152	1.033
cocoa price	1.187	-0.122	1.131	0.643	0.848
corn price	1.647	1.430	1.221	2.078	1.130
Feeder	0.993	1.172	1.930	1.383	1.336
Cattle price					
Fluid Milk	0.451	-0.122	-0.913	16.957	-7.556
price					
Lean Hogs	2.898	5.638	1.146	-0.442	0.959
price					
live cattle	1.090	-0.422	-0.700	2.933	5.459
price					
oats price	1.252	1.562	0.988	1.374	1.029
orange juice	0.781	0.618	1.172	1.164	0.753
price					
Pork Bellies	-1.016	2.651	2.015	-1.378	1.971
price					
Rapeseed	1.272	1.019	0.820	2.040	0.795
canola price					
Soybeans	1.676	1.147	1.160	-0.184	-0.362
price					
Sugar price	0.644	1.525	-7.519	1.880	1.314
wheat price	1.426	1.256	1.119	1.824	1.806
winnipeg	1.257	1.297	1.246	1.660	1.401
oats price					
Winnipeg	1.366	1.445	1.259	2.481	1.144
Wheat price					